

1 HARMONIC FUNCTIONS

Let $f(x, y) = u(x, y) + iv(x, y)$ be a complex valued function. If the second partial derivatives of u and v with respect to x and y exist and are continuous in the region R , then we find that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

This is done from the Cauchy-Riemann equations by simply differentiating them once again with respect to x and y and applying continuity. Functions such as $u(x, y)$ and $v(x, y)$ which satisfy this condition are called harmonic functions and are said to be harmonic in R .

Example 1.1 Prove that $e^{-x}(x \sin y - y \cos y)$ is harmonic and find v such that $f(z) = u + iv$ is analytic.

Solution

$$\frac{\partial u}{\partial x} = e^{-x} \sin y - e^{-x}(x \sin y - y \cos y) = e^{-x}(\sin y - x \sin y + y \cos y)$$

$$\frac{\partial^2 u}{\partial x^2} = -2e^{-x} \sin y + xe^{-x} \sin y - ye^{-x} \cos y.$$

$$\frac{\partial u}{\partial y} = e^{-x}(x \cos y + y \sin y - \cos y) = xe^{-x} \cos y + ye^{-x} \sin y - e^{-x} \cos y.$$

$$\frac{\partial^2 u}{\partial y^2} = -xe^{-x} \sin y + 2e^{-x} \sin y + ye^{-x} \cos y.$$

Hence $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ as required.

From the Cauchy-Riemann equations we have

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = e^{-x}(\sin y - x \sin y + y \cos y)$$

and

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -xe^{-x} \cos y - ye^{-x} \sin y + e^{-x} \cos y.$$

Now integrate the first one with respect to y and keep x constant to get

$$v = -e^{-x} \cos y + xe^{-x} \cos y + e^{-x}(y \sin y + \cos y) + F(x) = ye^{-x} \sin y + xe^{-x} \cos y + F(x).$$

Now substitute this into $\frac{\partial v}{\partial x}$ expression to get

$$-ye^{-x} \sin y - xe^{-x} \cos y + e^{-x} \cos y + F'(x) = -ye^{-x} \sin y - xe^{-x} \cos y - ye^{-x} \sin y$$

to get $F'(x) = 0$ so that $F(x) = c$ is a constant and hence

$$v = e^{-x}(y \sin y + x \cos y) + c.$$

Exercise 1: Show that the functions $u = 3x^2y + 2x^2 - y^3 - 2y^2$ and $u = xe^x \cos y - ye^x \sin y$ are harmonic and find their harmonic conjugates v .

Exercise 2: Define the term singularity as used in complex valued functions and explain the meaning of the following types of singularities: poles, removable singularity, essential, branch points and singularity at infinity. State an example of the same.